Tunneling Through the Opaque Inverted Parabola

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The inverted parabola remains an exactly solvable tunneling barrier when imaginary parts are added to the potential. They represent the effect of particle absorption. In the limit of strong absorption ("opaque barrier") convenient limiting forms for the tunneling probability are derived using asymptotic expansions of the exact scattering solution. An application to Zener tunneling is made and the result is compared to the case of tunneling under viscous dissipation. Significant differences are found between these two approaches to tunneling under dissipation.

1. Introduction

Quantum mechanical tunneling through the potential barrier

$$V(q) = -\tfrac{1}{2} m \omega^2 q^2 \tag{1}$$

at energies $E < 0$ is studied. For real values of E and ω, the exact result for the tunneling probability P is [1, 2]

$$P = [1 + \exp(-2\pi E/\hbar\omega)]^{-1}. \tag{2}$$

It will be demonstrated that even for complex values of E and ω a tunneling probability can be evaluated from an exact scattering solution. The physical interpretation of the imaginary parts in E and ω is particle absorption ("optical potential" [3]). The asymptotic expansion of the exact scattering solution is easy to handle in the case of strong absorption ("opaque barrier").

The theory is formulated in Section 2. Simple explicit results are obtained for either $\operatorname{Im} E \gg \operatorname{Re} E$, the case of strong uniform absorption treated in Sect. 3, or for parabolic absorption with $\operatorname{Im}\omega \gg \operatorname{Re}\omega$, discussed in Section 4. In Sect. 5 an application is made to the case of Zener tunneling, which also has the inverted parabola as tunneling barrier [4]. The results are compared to those for tunneling under viscous dissipation, as initiated by the work [5].

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2. Theory

The stationary Schrödinger equation associated with (1) is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} - m\omega^2 q^2/2\right)\psi = E\psi. \tag{3}$$

We seek a solution that corresponds to an incoming wave from the left, a reflected wave, and an outgoing wave.

Equation (3) is transformed by the substitutions

$$u = \sqrt{\frac{m\omega}{\hbar}}\, q\,, \qquad \varepsilon = \frac{E}{\hbar\omega}, \tag{4}$$

into

$$\frac{d^2\psi}{du^2} + u^2\psi + 2\varepsilon\psi = 0. \tag{5}$$

This equation has parabolic cylinder functions as solutions [6]. Consider the following one:

$$\psi = D_{i\varepsilon-1/2}((1-i)u). \tag{6}$$

In [7] this function is designated as

$$D_{i\varepsilon-1/2}(z) \equiv U(a = -i\varepsilon, z). \tag{6'}$$

Using formulae 9.246/1 and 9.246/2 in [6] one finds the following asymptotic expansions of ψ. For $u \ll -1$:

$$\psi \sim \exp(iu^2/2)[(1-i)u]^{i\varepsilon-1/2} + ((2\pi)^{1/2}i/\Gamma(\tfrac{1}{2}-i\varepsilon))$$
$$\cdot \exp(-iu^2/2 - \pi\varepsilon)[(1-i)u]^{-i\varepsilon-1/2} \tag{7}$$

and for $u \gg 1$:

$$\psi \sim \exp(iu^2/2)[(1-i)u]^{i\varepsilon-1/2}. \tag{7'}$$

The local momentum associated with each of the individual terms is mainly given by the gradient of the dominant phase in the exponentials. Thus the terms in (7) and (7′) mean in successive order: incident wave from the left, reflected wave back to the left and transmitted wave to the right of the barrier. The function D_p in (6) is, therefore, the uniformly valid scattering solution of our problem. It is not necessary to match wave functions at the classical turning points. The latter procedure is problematic for optical barriers due to non-conservation of wave number, particularly in the case of strong absorption.

We determine the tunneling probability recorded between spatial positions $-u$ and $u > 0$ according to the formula

$$P(u) \equiv |\psi_{\text{out}}(u)|^2 / |\psi_{\text{in}}(-u)|^2 . \tag{8}$$

For arbitrary complex ε (but real u), (7) and (7′) then give

$$P(u) = |\Gamma(1/2 - i\,\varepsilon)|^2 \exp(\pi \operatorname{Re}\varepsilon)(2u^2)^{-2\operatorname{Im}\varepsilon}/2\pi . \tag{9}$$

In the usual case of no absorption one has $\operatorname{Im}\varepsilon = 0$. One can then use the formula

$$|\Gamma(1/2 - i\,\varepsilon)|^2 \equiv \pi/\cosh\pi\varepsilon , \tag{10}$$

to obtain (2), irrespective of the reference position u.

We discuss the analytic continuation of (8) to complex values of ε and ω in more detail by considering two cases: (11)

i) $E \to \tilde{E} \equiv E + i\,W$; $\tilde{\varepsilon} \equiv \tilde{E}/\hbar\omega = \varepsilon + i\,W/\hbar\omega$,

ii) $\omega^2 \to \omega(\omega + i\,\gamma) \equiv \omega^2(1 + i\,v)$. (12)

In the first case, the potential (1) is altered according to

$$V \to V - i\,W . \tag{13}$$

The second case is then represented by

$$V \to V - i\,v\,m\,\omega^2 q^2/2 . \tag{14}$$

In the latter case, an additional transformation is necessary to map (3) onto (5), which amounts to

$$u \to (1 + i\,v)^{1/4} u \equiv w . \tag{15}$$

While E remains real in this case, the quantity $\tilde{\varepsilon}$ is, nevertheless, complex:

$$\tilde{\varepsilon} = E/(1 + i\,v)^{1/2}\hbar\omega . \tag{16}$$

Because of (15), (9) is not directly applicable to case ii).

From the continuity equation

$$\frac{\partial}{\partial t}|\psi|^2 + \frac{\partial}{\partial q}j = 2\left(\frac{\operatorname{Im}W}{\hbar}\right)|\psi|^2 \tag{17}$$

for the probability density $|\psi|^2$ and probability current density

$$j = \frac{\hbar}{2\,i\,m}\left[\psi^* \frac{\partial}{\partial q}\psi - \text{c.c.}\right], \tag{18}$$

it is seen, that $W > 0$ in (13) corresponds to uniform particle absorption everywhere in space. In contrast, (14) describes parabolic absorption. Since the partial waves are attenuated in their propagation directions, it is necessary to specify the space points between which tunneling is recorded, as already indicated in (8). The natural choice is

$$u^2 = u_1^2 \equiv -2\operatorname{Re}\tilde{\varepsilon} \equiv -2\,\varepsilon . \tag{19}$$

These are the locations of the classical turning points at energy $E < 0$.

3. Uniform Absorption

With the prescription (19) we encounter a difficulty in case i), that of uniform absorption: The order index p in the parabolic cylinder function (6), i.e.

$$p = i\tilde{\varepsilon} - \frac{1}{2} \equiv -a - \frac{1}{2} = -\left(\frac{W}{\hbar\omega} + \frac{1}{2}\right) - i\varepsilon , \tag{20}$$

also contains the quantity ε which is related to u_1 by (19). Isolation of incident and transmitted wave from the solution (6) requires asymptotically large values of u^2 and hence ε. For small $W < |E|$ this implies an asymptotic expansion of D_p with $|p| \approx u^2 \gg 1$. This is a very difficult region for asymptotic expansions. However, for strong absorption

$$W \gg |E| , \tag{21}$$

one finds $|p| \gg u^2$, even for $u^2 \gg 1$, and this asymptotic regime can be handled with some ease. We call it the "opaque" barrier. From [7], one finds for $a \approx |a| \gg u^2$: (22)

$$U(a, z = (1 - i)u) \sim c(a)\exp(-\sqrt{a}\,z), \quad u = u_1 \gg 1 ,$$

$$U(a, z = (1 - i)u) \sim c(a)\exp(-\sqrt{a}\,z), \quad u = -u_1 \ll -1 .$$

Thus we obtain for the tunneling probability according to (8)

$$P(u_1) = |U(a, (1 - i)\,u_1)/U(a, -(1 - i)\,u_1)|^2$$
$$\sim \exp(-4\sqrt{a}\,\operatorname{Re}[(1 - i)\,u_1]) . \tag{23}$$

Using (19) and $a \approx W/\hbar\,\omega$ one obtains explicitly:

$$P \sim \exp\left[-4 \left(\frac{W}{\hbar\,\omega} 2 \frac{|E|}{\hbar\,\omega} \right)^{1/2} \right]. \quad (24)$$

A considerable distortion of the original tunneling formula (2) in the corresponding limit of $|E| \gg \hbar\,\omega$ (WKB-limit), namely

$$P_{WKB} = \exp\left(-2\pi |E|/\hbar\,\omega \right), \quad (25)$$

can thus be noted, when strong uniform absorption occurs. Since $W \gg |E|$, the tunneling probability (25) is considerably smaller than P_{WKB}, as has to be expected.

4. Parabolic Absorption

We now turn to case ii) described by (12) and (14–16). The exact scattering solution reads explicitly:

$$\psi = D_{-1/2+i\bar{\varepsilon}}((1+i\,v)^{1/4}\,u) \equiv D_{-1/2+i\varepsilon}(w). \quad (26)$$

Under strong absorption, the inequality

$$|w_1| \equiv |(1+i\,v)^{1/4}\,u_1| \gg \left| -\frac{1}{2} + \frac{i\,\varepsilon}{(1+i\,v)^{1/2}} \right|$$

is valid because of $v \gg 1$. Using (7), (7′) and (8) with w in place of u, one finds

$$P(u_1) \sim \left| \Gamma\left(\frac{1}{2} + (1+i)\frac{|\varepsilon|}{\sqrt{2\,v}} \right) \right|^2$$

$$\cdot (2\,|\varepsilon|^{1/2}\,v^{1/4})^{-4|\varepsilon|/\sqrt{2\,v}}$$

$$\cdot \exp\left(-\frac{\pi}{2}\,|\varepsilon|/\sqrt{2\,v} - 2\sqrt{2}\,|\varepsilon|\,v^{1/2} \right)/2\,\pi$$

$$\to \frac{1}{2} \exp(-2\sqrt{2}\,|\varepsilon|\,v^{1/2}). \quad (27)$$

It is noted that the reflection coefficient defined by

$$R(u_1) = \frac{|\psi_{ref}(-u_1)|^2}{|\psi_{in}(-u_1)|^2} \quad (28)$$

turns out to be identical to $P(u_1)$ in this particular case. Since particle conservation does not hold any more, the usual relation $P + R = 1$ does not apply.

5. An Application:
Zener Tunneling under Dissipation

The inverted parabola as tunneling barrier appears in several physical applications, e.g. pair creation due to an homogeneous electric field in Klein-Gordon- and Dirac-particle-equations [8] and in Zener tunneling [4]. In the case of relativistic particles, the "opaque" limit makes no sense since it corresponds to particle life times much smaller than $\hbar/E_0$, where E_0 is the rest energy.

It is, however, interesting to study Zener tunneling in the present context. Using a Euclidean description of tunelling [9] it can be shown [4] that Zener tunneling in an insulator (and in the usual WKB limit) is represented by tunneling through the inverted parabola using the correspondencies:

$$E \leftrightarrow -V_B \equiv -\varepsilon_g^2/16\,\varepsilon_F,$$

$$\omega \leftrightarrow \omega(F) \equiv e_0 F/\hbar\,k_F. \quad (29)$$

Here, ε_g is the energy gap, $\varepsilon_F = \hbar\,k_F\,v_F/2$ the Fermi energy, $\hbar\,k_F$ Fermi momentum, and v_F Fermi velocity (in the absence of the gap). The charge of the tunneling carriers is e_0 and the electric field strength is F. Substituting (29) into (25) gives Zener's well known result [10] for the tunneling probability

$$P = \exp(-\pi\,\varepsilon_g^2/4\hbar\,v_F\,e_0\,F). \quad (30)$$

Using our results of Sect. 3 and 4 we can investigate Zener tunneling under absorption in the limit

$$\varepsilon_F \gg W \gg V_B. \quad (31)$$

We will also compare the results to those for tunneling under viscous dissipation.

The modern theory of tunneling under viscous dissipation originated from the work [5]. From this work, an estimate can be found for the reduction factor in the WKB tunneling probability under viscous dissipation:

$$\Delta P \approx \exp(-\eta(\Delta q)^2/\hbar), \quad (32)$$

when the tunneling particle experiences a viscous force $F = -\eta\,\dot{q}$. The distance under the barrier is Δq. Formula (3) can be understood as a Boltzmann factor for two competing energies, the classical energy loss in tunneling and the quantum mechanical energy fluctuation − both associated with the tunneling motion in the inverted barrier, i.e. taken in the Euclidean sense [9].

We formally relate the absorption constant W in (11) to a classical viscosity η. W causes a time decay in the wave function with decay constant

$$\tau = \hbar/W. \quad (33)$$

The viscosity is thus

$$\eta \equiv m/\tau = m\, W/\hbar\,, \qquad (34)$$

or

$$W/\hbar\omega \equiv 1/\omega\,\tau = \eta/m\,\omega \equiv \alpha\,. \qquad (35)$$

Application of (24) – requiring $\alpha \gg 1$ – to Zener tunneling thus gives immediately

$$P \sim \exp(-4\sqrt{\alpha/\pi}\,\sqrt{\pi\,\varepsilon_g^2/4\hbar\,v_F\,e_0\,F})\,. \qquad (36)$$

Equation (32) together with (30) predicts

$$P \approx \exp(-(4\alpha + \pi)\,\varepsilon_g^2/4\hbar\,v_F\,e_0\,F)\,. \qquad (37)$$

In [4], a result for collective Zener tunneling under viscous dissipation was derived which differs both from (36) and (37). This is, however, due to the particular dependence of the effective viscosity used there, on the renormalized "bounce time" [9] in the inverted potential. A strictly constant viscosity would also lead to (37), in accord with the general prediction (32).

We must thus conclude that the opaque barrier does not provide a proper description of tunneling under strong viscous dissipation.

For completeness, we also consider case ii) in the present context:

An effective α as function of v can be defined by

$$\alpha = \frac{1}{2}\,v\,\frac{m\,\omega}{\hbar}\,q_1^2 \equiv \frac{1}{2}\,v\,u_1^2\,. \qquad (38)$$

Thus one finds from (27)

$$\begin{aligned}
P &\sim \frac{1}{2}\exp\left(-4\,|\varepsilon|\,\frac{\sqrt{\alpha}}{u_1}\right)\\
&\equiv \frac{1}{2}\exp(-4\sqrt{\alpha/\pi}\,\sqrt{\pi\,\varepsilon_g^2/4\hbar\,v_F\,e_0\,F})\,,
\end{aligned} \qquad (39)$$

which is about the result (36). For practical purposes there is thus not much difference between case i) and case ii) in the limit of strong absorption.

6. Summary

The tunneling properties of the inverted parabola with strong absorption ("opaque" barrier) have been studied using the exact scattering solution available for this model. The tunneling probability is found to decrease strongly and its functional form is considerably changed compared to the case without absorption. A comparison with available results on Zener tunneling under viscous dissipation shows that an "optical" barrier is not a proper means to describe this type of dissipation mechanism.

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